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Invariant Subspaces of Clustering Operators

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A class of clustering operators is defined which is a generalization of a transfer matrix of a Gibbs lattice field with an exponential decay of correlations. It is proved that for small values of β the clustering operator has invariant subspaces which are similar to k-particle subspaces of the Fock space. The restriction of the clustering operator onto these subspaces resembles the operator $\exp(-H_k)$, where H_k is the k-particle Schrödinger Hamiltonian in nonrelativistic quantum mechanics. The spectrum of each H_k, $k \geqslant 1$, is contained in the interval $(C_1\beta^k, C_2\beta^k)$. These intervals do not intersect with each other.

KEY WORDS: Clustering operators; clustering estimates; transfer matrix; spectrum of many-body system; one-particle space; two-, three-, and many-particle spaces.

1. INTRODUCTION

Recent work in statistical physics and quantum field theory suggests the significance of the following class of operators, which appear naturally in many-body systems with an exponential decay of correlations.

Let $\mathfrak{A}$ be the set of all finite subsets (including the empty one) of $\mathbb{Z}^\nu$; μ is the measure $\mu(B) = |B|$ for $B \in \mathfrak{A}$.

An operator A in $L_2(\mathfrak{A}, \mu)$

$$(Af)(T) = \sum_{T' \in \mathfrak{A}} a_{T,T'} f(T') \tag{1.1}$$

is called clustering if:

1. A commutes with translations U_t, $t \in \mathbb{Z}^\nu$, in $L_2(\mathfrak{A}, \mu)$,

$$(U_t f)(T) = f(T - t) \tag{1.2}$$

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2. $a_{\varnothing,\varnothing} = 1$, $a_{\varnothing,T} = 0$ if $T \neq \varnothing$ and for $T, T' \neq \varnothing$ one has

$$a_{T,T'} = \sum_{k=1,2,\dots} \sum_{\{\tau_1,\dots,\tau_k\}} \omega_k(\tau_1,\dots,\tau_k) \tag{1.3}$$

where $\tau_i = (T_i, T_i') \in \mathfrak{A} \times \mathfrak{A}$, $T_i, T_i' \neq \varnothing$, $\tau_1 \cup \tau_2 = (T_1 \cup T_2, T_1' \cup T_2')$, and the same for intersections, $\hat{\varnothing} = (\varnothing, \varnothing)$. The summation in (1.3) is over all (nonordered) partitions

$$(T, T') = \tau_1 \cup \cdots \cup \tau_k, \qquad \tau_i \cap \tau_j = \hat{\varnothing} \qquad \text{if} \quad i \neq j$$

It is assumed that "cluster functions" $\omega_k(\tau_1,\dots,\tau_k)$ are symmetric with respect to permutations and satisfy the following conditions:

1. For all $k = 1, 2,\dots$, and for any $s_1,\dots, s_k \in \mathbb{Z}^\nu$

$$\omega_k(\tau_1 + s_1,\dots, \tau_k + s_k) = \omega_k(\tau_1,\dots,\tau_k) \tag{1.4}$$

where $\tau + s = (T + s, T' + s)$ if $\tau = (T, T')$. Although in fact ω_k in (1.3) depends only on collections with $\tau_i \cap \tau_j = \hat{\varnothing}$, $i \neq j$, one can assume that ω_k are defined for all collections $(\tau_1,\dots,\tau_k)$.

2. There exist β, $0 < \beta < 1$ (clustering parameter), and $M > 0$ such that for any $k = 1, 2,\dots$

$$|\omega_k(\tau_1,\dots,\tau_k)| < M(\beta)^\eta, \qquad \eta = \sum_{i=1}^{k} d_{\tau_i} \tag{1.5}$$

where d_τ is defined in the following way. Let us identify $\mathbb{Z}^\nu$ with

$$Y_0 = \{t \in \mathbb{Z}^{\nu+1}: \quad t = (t^1,\dots, t^\nu, 0), t^i \in \mathbb{Z}^1, i = 1,\dots,\nu\}$$

We write also

$$Y_k = \{t \in \mathbb{Z}^{\nu+1}: \quad t = (t^1,\dots, t^\nu, k), t^i \in \mathbb{Z}^1, i = 1,\dots,\nu\}$$

For any $t \in \mathbb{Z}^\nu$ ($t \in Y_0$) we write $t(k) = t + ke_{\nu+1}$, $e_{\nu+1} = (0,\dots, 0, 1) \in \mathbb{Z}^{\nu+1}$. In a similar way we can define $T(k)$ for any $T \subset \mathbb{Z}^\nu$.

Then d_B, $B \subset \mathbb{Z}^{\nu+1}$, is defined to be the minimum length of a tree, the vertices of which are the points of B. The length of a tree is the sum of the lengths of its lines, and the length of a line is taken in the metric

$$\rho(t_1, t_2) = \sum_{i=1}^{\nu+1} |t_1^i - t_2^i|, \qquad t_j = (t_j^1,\dots, t_j^{\nu+1}), \quad j = 1, 2$$

We define

$$d_\tau = d_{T \cup T'(1)}$$

where $\tau = (T, T')$ and $T \subset Y_0$, $T'(1) \subset Y_1$.

Remark. We note that the subspace

$$L_0 = \{f(T): \quad f(T) = 0 \quad \text{if } T \neq \varnothing\}$$

and its orthogonal complement $L_+ \subset L_2(\mathfrak{A})$ are invariant with respect to A. One can show that if the representation (1.3) exists and satisfies (1.4) and (1.5), then it is unique. The adjoint operator A^* is also clustering and its cluster functions are equal to $\overline{\omega_k(\tau_1{}^*, \ldots, \tau_k{}^*)}$, $\tau^* = (T', T)$ if $\tau = (T, T')$. In particular, a clustering operator A is self-adjoint iff

$$\omega_k(\tau_1, \ldots, \tau_k) = \overline{\omega_k(\tau_1{}^*, \ldots, \tau_k{}^*)}$$

Let $L^n \subset L_2(\mathfrak{A})$, $n \geqslant 1$, be the subspace of functions $f(B)$ with $f(B) = 0$ when $|B| \neq n$. Then

$$L_+ = \bigoplus_{n=1}^{\infty} L^n \tag{1.6}$$

An operator A acting in L^n is called clustering iff an operator in $L_2(\mathfrak{A})$ equal to A on L^n and to 0 on $(L^n)^\perp$ is clustering.

The main result of this paper is the following:

Theorem 1.1. Let a self-adjoint clustering operator A be given such that for each $k = 1, 2, \ldots$ and any $t_1, \ldots, t_k$

$$\omega_k(\tau_{t_1}, \ldots, \tau_{t_k}) > L(C\beta)^k \tag{1.7}$$

where $\tau_t = (\{t\}, \{t\})$, $t \in \mathbb{Z}^\nu$, and L and C do not depend on $k, t_1, \ldots, t_k$.

Then for each $N \geqslant 1$ there exists $\beta_0 = \beta_0(N, M, C, L)$ such that for $0 < \beta < \beta_0$ there exists $N + 1$ mutually orthogonal subspaces $\mathscr{H}_1, \ldots, \mathscr{H}_N$, $\mathscr{H}_{N+1} \subset L_+$ such that they are invariant with respect to A and U_t and for $1 \leqslant s \leqslant N$

$$\begin{aligned} K_1\beta^s\|x\|^2 \leqslant (Ax, x) \leqslant K_2\beta^s\|x\|^2, \qquad & x \in \mathscr{H}_s \\ |(Ax, x)| < K_2\beta^{N+1}\|x\|^2, \qquad & x \in \overline{\mathscr{H}}_{N+1} \end{aligned} \tag{1.8}$$

where $K_i = K_i(N, M, C, L)$ are constants, $i = 1, 2$. $\overline{\mathscr{H}}_{N+1}$ is the orthogonal complement of $\mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_N$.

Moreover, $\mathscr{H}_s$, $1 \leqslant s \leqslant N$, lies in the vicinity of L^s, i.e.,

$$\|x - P_{L^s}x\| \leqslant G\beta^{1/2}\|x\|, \qquad x \in \mathscr{H}_s \tag{1.9}$$

where P_{L^s} is the projection onto L^s, and $G = G(N, M, C, L)$ is a constant. For each $k = 1, \ldots, N$ there exists a unitary operator $V_k: \mathscr{H}_k \to L^k$ such that

1. $V_k(U_k|\mathscr{H}_k)V_k{}^{-1} = U_t|_{L^k}$
2. The operator

$$\tilde{A}_k = V_k(A|\mathscr{H}_k)V_k^{-1}$$

in L^k is clustering with clustering parameter $\lambda(\beta) \to 0$ when $\beta \to 0$.

It is natural to call $\mathscr{H}_k$ a k-point subspace, as A is clustering and is similar to $\exp(-\lambda H)$, where H is the k-particle Schrödinger Hamiltonian.

This leads us to conjecture that the spectrum of A in $\mathscr{H}_k$ can be studied with known methods.[7,8]

In this paper we prove the first part of Theorem 1.1, i.e., the existence of invariant subspaces and formulas (1.8) and (1.9). The last assertion of this theorem will be proved elsewhere.

It is proven in Ref. 4 that the conditions of Theorem 1.1 are satisfied for the transfer matrix of the Ising model for arbitrary dimension $\nu \geqslant 2$ and for sufficiently small β. To study the spectrum of transfer matrices for more general random fields which can take arbitrary values our definition of the clustering operator can be generalized. For the Ising model on $\mathbb{Z}^{\nu+1}$, e.g., $w_k(\tau_i,..., \tau_k)$ is equal to $\prod_{i=1}^{k} w_1(T_i)$ and $w_1(T)$ is the semiinvariant of the random variables $\hat{\sigma}_t$, $t \in T \subset \mathbb{Z}^\nu$ (zero time slice), where $\hat{\sigma}_t$ is the spin variable σ_t minus the orthogonal projection of σ_t onto the subspace $L_2(\Sigma_{0,t})$ and $\Sigma_{0,t}$ is the minimal σ-algebra with respect to which all σ_t are measurable for $t^1 \subset \mathbb{Z}^\nu$ (zero time) and less than t in lexicographic order (see Ref. 4).

2. CONSTRUCTION OF INVARIANT SUBSPACES

The constants which appear in subsequent considerations depend on N, M, and L.

Lemma 2.1. Let A be an operator for which the conditions of Theorem 1.1 are satisfied. Then for each N there exists β_0 such that for $0 < \beta < \beta_0$ and any s, $1 \leqslant s \leqslant N$, there exists a representation

$$L_+ = \mathscr{L}^s \oplus \bar{\mathscr{L}}^s$$

where both $\mathscr{L}^s$ and $\bar{\mathscr{L}}^s$ are invariant with respect to A and

$$(Ax, x) > K_1\beta^s\|x\|^2, \qquad x \in \mathscr{L}^s \tag{2.1a}$$

$$|(Ax, x)| < K_2\beta^{s+1}\|x\|^2, \qquad x \in \bar{\mathscr{L}}^s \tag{2.1b}$$

for some constants K_1 and K_2.

The proof of this lemma will be given below; now we shall deduce Theorem 1.1 from it.

We can assume of course that $K_1 > K_2\beta$ and then $K_1\beta^s > K_2\beta^{s+1}$. From this and from (2.1a) and (2.1b) it follows that

$$Q_{\mathscr{L}^s} = E_{(K_1\beta^s,\infty)}, \qquad Q_{\bar{\mathscr{L}}^s} = E_{(-K_2\beta^{s+1},K_2\beta^{s+1})}$$

where $Q_{\mathscr{L}^s}$ and $Q_{\bar{\mathscr{L}}^s}$ are projections onto $\mathscr{L}^s$ and $\bar{\mathscr{L}}^s$, respectively, and E_Δ, $\Delta \subset \mathbb{R}^1$, is the spectral family for A. In particular it follows that

$$\mathscr{L}^1 \subseteq \mathscr{L}^2 \subseteq \cdots \subseteq \mathscr{L}^N$$

Let us put

$$\mathscr{H}_1 = \mathscr{L}^1, \qquad \mathscr{H}_2 = \mathscr{L}^2 \ominus \mathscr{L}^1, \dots, \qquad \mathscr{H}_N = \mathscr{L}^N \ominus \mathscr{L}^{N-1}, \qquad \mathscr{H}_{N+1} = \bar{\mathscr{L}}^N$$

One can easily verify that these subspaces are invariant and satisfy (1.8). Later we shall prove (1.9).

Proof of Lemma 2.1. Let us denote

$$L^{\leqslant n} = \bigoplus_{k=1}^{n} L^k, \qquad L^{>n} = \bigoplus_{k=n+1}^{\infty} L^k$$

Each operator B acting in L_+ can be represented as an operator matrix $B \sim \|B_{mn}\|$, $m, n = 1, 2, \dots$, in the representation (1.6), where $B_{mn} = P_{L^m} B P_{L^n}$. Similarly, the decomposition $L_+ = L^{\leqslant k} \oplus L^{>k}$ induces the operator matrix

$$B = \begin{pmatrix} B_{11}^k & B_{12}^k \\ B_{21}^k & B_{22}^k \end{pmatrix}$$

Moreover,

$$B_{11}^k \sim \|B_{mn}\|, \qquad m, n \leqslant k$$

and similarly for the rest of the $B_{ij}^{(k)}$.

Let us consider the diagonal operator

$$(Hf)(T) = h_T f(T)$$

with

$$h_T = \omega_k(\tau_{t_1}, \dots, \tau_{t_k})$$

where τ_t is defined above, and $T = \{t_1, \dots, t_k\}$. Let us put $\hat{A} = A - H$ and let $\hat{a}_{T,T'}$ be the matrix elements of $\hat{A}$.

Lemma 2.2. Let a clustering operator A be given and let its cluster functions ω_k satisfy (1.5) with $\beta > 0$ sufficiently small. Then for any T and any $k \geqslant 1$

$$\sum_{T':|T'|=k} |\hat{a}_{T,T'}| \leqslant \begin{cases} (D\beta)^{\max(k,m)}, & k \neq m \\ (D\beta)^{m+1}, & k = m \end{cases} \tag{2.2}$$

where $m = |T|$, and D depends only on ν.

Corollaries to Lemma 2.2

1. If $\hat{A} \sim (\hat{A}_{mn})$, then

$$\|\hat{A}_{mn}\| \leqslant \begin{cases} (D\beta)^{\max(m,n)}, & m \neq n \\ (D\beta)^{m+1}, & m = n \end{cases} \tag{2.3}$$

In fact, for $m \neq n$

$$\|\hat{A}_{mn} + \hat{A}_{nm}\| = \|\hat{A}_{mn}\| = \|\hat{A}_{nm}\|$$

and due to the self-adjointness of $\hat{A}_{mn} + \hat{A}_{nm}$

$$\|\hat{A}_{mn} + \hat{A}_{nm}\| = \sup_{f \in L_2, \|f\| = 1} |((\hat{A}_{mn} + \hat{A}_{nm})f, f)|$$

$$= 2 \sup_{\|f\|=1} \left| \sum_{T, T': |T| = M, |T'| = n} \hat{a}_{T',T} f(T) f(T') \right|$$

$$\leqslant \sup_{\|f\|=1} \sum_{|T| = m, |T'| = n} |\hat{a}_{T',T}|(|f(T)|^2 + |f(T')|^2)$$

$$\leqslant \sup_{\|f\|=1} \left\{ \sup_{|T| = m,} \sum_{|T'| = n} |\hat{a}_{T',T}| \sum_{|\tilde{T}| = m} |f(\tilde{T})|^2 \right.$$

$$\left. + \sup_{|T'| = n} \sum_{|T| = m} |\hat{a}_{T',T}| \sum_{|\tilde{T}| = n} |f(\tilde{T}')|^2 \right\} \leqslant (D\beta)^{\max(m,n)}$$

The case $m = n$ can be examined similarly.

2. As $\|H_{nn}\| \leqslant M\beta^n$, we get for all m, n

$$\|A_{mn}\| \leqslant (\tilde{D}\beta)^{\max(m,n)} \tag{2.4}$$

for some $\tilde{D}$. Here $(A_{mn}) \sim A$.

3. Using the bound

$$\|Q\| \leqslant \sup_m \sum_n \|Q_{mn}\| \tag{2.5}$$

for any $Q \sim (Q_{mn})$ in L_+ $(Q^* = Q)$ and the bound (2.4), one can find that

$$\max\{\|A_{22}^k\|, \|A_{21}^k\| = \|A_{12}^k\|\} \leqslant B(\tilde{\tilde{D}}\beta)^{k+1} \tag{2.6}$$

for some B, $\tilde{\tilde{D}}$ and all $k = 1, 2, \ldots$.

4. Using the previous bounds one can find that for all $k = 1, 2, \ldots$

$$\max\{|(A_{12}^k x, y)|, |(A_{22}^k x, y)|, |(A_{21}^k x, y)|\} \leqslant B(\tilde{\tilde{D}}\beta)^{k+1}\|x\| \cdot \|y\|, \quad x, y \in L_+ \tag{2.7}$$

and for some constant R

$$|(\hat{A}_{11}^k x, x)| < R(D\beta)^{1/2} \sum_{s=1}^{k} (D\beta)^s \|P_{L^s} x\|^2 \tag{2.8}$$

5. Using the bound $h_T > L(C\beta)^{|T|}$ and (2.8) one can get, for all k such that

$$(2R/L)(D/C)^k (D\beta)^{1/2} < 1 \tag{2.9}$$

the following inequality:

$$(Ax, x) > (L/2)(C\beta)^k \|x\|^2, \qquad x \in L^k \tag{2.10}$$

Lemma 2.3. If the clustering operator A satisfies (1.7), then for any $N \geqslant 1$ there exists $\beta_0 = \beta_0(N)$ such that for all $\beta < \beta_0$ and $1 \leqslant k \leqslant N$ the operator A_{11}^k is invertible in $L^{\leqslant k}$ and for any $m, n \leqslant k$

$$\|(A_{11}^k)_{mn}^{-1}\| < \tilde{L}\beta^{\max(m,n)-m-n} \tag{2.11}$$

where $\tilde{L}$ is a constant.

Corollary to Lemma 2.3. From (2.11) and (2.5) one can find that

$$\|(A_{11}^k)^{-1}\| < \tilde{\tilde{L}}\beta^{-k} \tag{2.12}$$

where $\tilde{\tilde{L}}$ is a constant [cf. (2.9)].

Proofs of Lemmas 2.2 and 2.3 will be given in the next section; now we shall derive Lemma 2.1 from them.

Proof of Lemma 2.1. We shall look for $\mathscr{L}^k$, $1 \leqslant k \leqslant N$, as a graph

$$\mathscr{L}^k = \{\psi + S^k\psi: \quad \psi \in L^{\leqslant k}\}$$

of some operator $S^k: L^{\leqslant k} \to L^{>k}$.

From $A\mathscr{L}^k \subseteq \mathscr{L}^k$ we get the following equation for S^k:

$$S^k = A_{21}^k(A_{11}^k)^{-1} + A_{22}^k S^k (A_{11}^k)^{-1} - S^k A_{12}^k S^k (A_{11}^k)^{-1}. \tag{2.13}$$

We can consider the right-hand side of this equation as the mapping G^k of the space $\mathfrak{A}_{L^{\leqslant K}, L^{>K}}$ of bounded operators from $L^{\leqslant k}$ into $L^{>k}$ into itself. We shall sometimes identify these operators with operators acting in L_+ and specified by the matrix

$$\begin{pmatrix} 0 & 0 \\ S^k & 0 \end{pmatrix}$$

with respect to the decomposition $L_+ = L^{\leqslant k} + L^{>k}$.

From (2.6) and (2.10) it follows that for sufficiently small β the mapping G^k leaves some sphere

$$\mathfrak{N} = \{\|S^k\| < \kappa\beta\} \subset \mathfrak{A}_{L^{\leqslant K}, L^{<K}} \tag{2.14}$$

fixed, where κ is a constant, and is a contraction in this sphere. It follows that there exists a solution of (2.13), i.e., there exists an invariant subspace $\mathscr{L}^k$. Then (2.1a) follows from (2.7) and (2.10) if we suppose that (2.9) holds for given N.

To prove (2.1b) we need the following result:

Lemma 2.4. For β sufficiently small and for all $1 \leqslant k \leqslant N$

$$\|S_{mn}^k\| < \bar{L}(\bar{D}\beta)^{m-k}, \qquad m > k, \quad n \leqslant k \tag{2.15}$$

where $\bar{D}$ and $\bar{L}$ are constants.

The proof of this lemma will be given below; now we shall prove (2.1b) using (2.15).

Let us note that $\bar{\mathscr{L}}^k$ is a graph of $(-S^k)^*$

$$\bar{\mathscr{L}}^k = \{\varphi - (S^k)^*\varphi: \quad \varphi \in L^{>k}\}$$

From this, for $x = \varphi - (S^k)^*\varphi \in \bar{\mathscr{L}}^k$, $\varphi \in L^{>k}$, we have

$$(Ax, x) = (A_{22}^k\varphi, \varphi) - (A_{12}^k\varphi, (S^k)^*\varphi) - (A_{21}^k(S^k)^*\varphi, \varphi) + (A_{11}^k(S^k)^*\varphi, (S^k)^*\varphi)$$

By (2.7)

$$\max\{|(A_{22}^k\varphi, \varphi)|, |(A_{12}^k\varphi, (S^k)^*\varphi)|, |(A_{22}^k(S^k)^*\varphi, \varphi)|\} < \tilde{K}_2\beta^{k+1}\|\varphi\|^2 < \tilde{K}_2\beta^{k+1}\|x\|^2 \tag{2.16}$$

Moreover, one can easily get a bound

$$|(A_{11}^k(S^k)^*\varphi, (S^k)^*\varphi)| < \sup_{m>k,n>k} \sum \|(S^kA_{11}^k(S^k)^*)_{m,n}\| \cdot \|\varphi\|^2 \tag{2.17}$$

Using

$$\|(S^kA_{11}^k(S^k)^*)_{m,n}\| \leqslant \sum_{t,p} \|S_{m,t}^k\| \, \|(A_{11}^k)_{t,p}\| \, \|(S^k)_{p,n}^*\|$$

and (2.4) and (2.15), we get

$$\|(S^kA_{11}^k(S^k)^*)_{mn}\| \leqslant \hat{R}_0(\hat{D}_0\beta)^{m+n-k}$$

for some constants $\hat{R}_0$ and $\hat{D}_0$. From this inequality and from (2.17) we get

$$|(A_{11}^k(S^k)^*\varphi, (S^k)^*\varphi)| < L\beta^{k+2}\|\varphi\|^2 \tag{2.18}$$

Using (2.16) and (2.18), one can prove (2.1b). Lemma 2.1 is proved.

Proof of Lemma 2.4. Let us fix some constant $\bar{D}$ and consider a set $\mathfrak{B}_{\bar{D}}{}^k \subset \mathfrak{A}_{L^{\leqslant k}, L^{>k}}$ of operators from $L^{\leqslant k}$ into $L^{>k}$ and such that their matrix elements $S \sim (S_{mn})_{m>k; n\leqslant k}$ satisfy the following inequality

$$\|S_{mn}\| < K(\bar{D}\beta)^{m-n}, \qquad m > k, \quad n \leqslant k \tag{2.19}$$

with some constant K.

The space $\mathfrak{B}_{\bar{D}}{}^k$ is complete with respect to the norm

$$\|S\|_{\bar{D}} = \sup_{m>k, n\leqslant k} \{\|S_{m,n}\|(\bar{D}\beta)^{-(m-n)}\} \tag{2.20}$$

Lemma 2.4 follows from the following:

Lemma 2.5. For $\bar{D} = \bar{D}(C, N)$ sufficiently large and β sufficiently small the mapping G^k satisfies the following conditions:

1. It maps $\mathfrak{B}_{\bar{D}}{}^k$ into itself.
2. It leaves some sphere $\hat{\mathfrak{R}} \subset \mathfrak{B}_{\bar{D}}{}^k$,

$$\hat{\mathfrak{R}} = \{\|S\|_{\bar{D}} < \hat{\kappa}\beta\}$$

fixed, where $\hat{\kappa}$ is a constant.

3. It is a contraction in $\hat{\mathfrak{R}}$.

The proof of this lemma can be obtained from (2.3) and (2.11).

Proof of (1.11). From the decomposition

$$x = x_{<k} + x_k + x_{>k} = \varphi + S^k\varphi$$

where $x \in \mathscr{H}_k$, $x_k \in L^k$, $x_{<k} \in L^{\leqslant(k-1)}$, $x_{<k} \in L^{>k}$, and $\varphi \in L^{\leqslant k}$ one can see that

$$\varphi = x_{<k} + x_k, \qquad S^k\varphi = x_{>k}$$

So

$$\|x_{>k}\| < \|S^k\| \, \|\varphi\| \leqslant \|S^k\| \, \|x\| < \text{const} \times \beta\|x\| \tag{2.21}$$

Then it follows from (2.7) and (2.14) that

$$\begin{aligned}|(Ax, x) - (A\varphi, \varphi)| < {} & |(A_{12}^k S^k\varphi, \varphi)| + |(A_{12}^k\varphi, S^k\varphi)| \\ & + |(A_{22}^k S^k\varphi, S^k\varphi)| < K(D\beta)^{k+2}\|\varphi\|^2\end{aligned}$$

and from (1.8) we get

$$(A\varphi, \varphi) < K_2\beta^k\|\varphi\|^2$$
$$(A\varphi, \varphi) = (A_{11}^k x_{<k}, x_{<k}) + (A_{22}^k x_k, x_k) + (A_{21}^k x_{<k}, x_k) + (A_{12}^k x_k, x_{<k})$$

Also from (2.7) we get

$$|(A\varphi, \varphi) - (Ax_{<k}, x_{<k})| < (\tilde{D}\beta)^k\|x_k\| \, \|x_{<k}\|$$

and

$$|(Ax_{<k}, x_{<k})| < R_1\beta^k\|\varphi\|^2$$

where R_1 is a constant. Using (2.9), we get

$$\|x_{<k}\| < (R_1\beta)^{1/2}\|x\| \tag{2.22}$$

3. PROOF OF LEMMAS 2.2 AND 2.3

Let A be a clustering operator with matrix elements $a_{T,T'}$, and let $\omega_k(\tau_1,\dots,\tau_k)$ be its cluster functions, which satisfy a somewhat weaker condition than (1.6):

$$|\omega_k(\tau_1,\dots,\tau_k)| < M\prod_{i=1}^{k} \kappa^{|\tau_i|}(\lambda)^{d\tau_i} \tag{3.1}$$

where $\kappa > 0$, $0 < \lambda < 1$, $M > 0$ are constants and $|\tau| = |T| + |T'|$ if $\tau = (T, T')$.

Lemma 3.1. Let A be a self-adjoint clustering operator satisfying (3.1). Then for λ sufficiently small

$$\sum_{T':|T'|=k} |\hat{a}_{T,T'}| < (C\kappa)^{k+m}\begin{cases}(D\lambda)^{\max(k,m)}, & k \neq m \\ (D\lambda)^{m+1}, & k = m\end{cases} \tag{3.2}$$

where $m = |T|$, $\hat{a}_{T,T'}$ represents the matrix elements of $\hat{A}$, and C and D are some constants.

Proof. Let $|T| = m$. We consider the case $k \neq m$. Then

$$\sum_{T'=k} |\hat{a}_{T,T'}| < M\kappa^{m+k} \sum_{|T'|=k} \sum_{\substack{(\tau_1,\ldots,\tau_s) \\ \cup\tau_i=(T,T') \\ s=1,2,\ldots,\min(k,n)}} \prod_{i=1}^{s} (\lambda)^{d_{\tau_i}}$$

Let us denote for $T \neq \varnothing$ and for $s = 1, 2,\ldots$

$$\tilde{\omega}_s^{\lambda}(T) = \sum_{\hat{T}:|\hat{T}|=s} (\lambda)^{d_{(T\cup\hat{T}(1))}}$$

Then

$$\sum_{|T'|=k} |a_{T,T'}| < M\kappa^{m+k} \sum_{p=1}^{\min(k,m)} (1/p!) \sum_{\substack{k_1,\ldots,k_p \\ k_1+\ldots+k_p=k}} \sum_{(T_1,\ldots,T_p):T_1\cup\ldots\cup T_p=T} \prod_{i=1}^{p} \tilde{\omega}_{k_i}^{\lambda}(T_i) \tag{3.3}$$

Here $\sum_{k_1,\ldots,k_p}$ means summation over all ordered collections of integers k_i, $i = 1,\ldots, p$, such that $\sum_{i=1}^{p} k_i = k$. In $\sum_{T_1,\ldots,T_p}$ the summation is over all ordered collections of mutually disjoint $T_i \subseteq T$, $i = 1,\ldots, p$, the union of which is equal to T.

Further,

$$\sum_{(T_1,\ldots T_p):\cup T_i=T} \prod_{i=1}^{p} \tilde{\omega}_{k_i}^{\lambda}(T_i) < \sum_{(p_1,\ldots,p_p)} \prod_{i=1}^{p} \gamma_{p_i,k_i}^{\lambda}(T)$$

where

$$\gamma_{l,k}^{\lambda}(T) = \sum_{\tilde{T}\subset T,|\tilde{T}|=l} \tilde{\omega}_k^{\lambda}(\tilde{T}) < \sum_{\substack{\tilde{T}\subset T,\hat{T}\subset\mathbb{Z}^{\nu} \\ |\tilde{T}|=l,|\hat{T}|=k}} \lambda^{d_{\tilde{T}\cup\hat{T}(1)}}$$

$$< |T|\left(\hat{C} \sum_{0\neq x\in\mathbb{Z}^{\nu+1}} \lambda^{d(0,x)}\right)^{l+k-1} \tag{3.4}$$

where $\hat{C}$ is a constant. The last bound can be obtained as in Refs. 5 and 6. Further, for $\lambda < 1/2$

$$\sum_{0\neq x\in\mathbb{Z}^{\nu+1}} \lambda^{d(0,x)} < C_{\nu}\lambda \tag{3.5}$$

where $C_{\nu} > 1$ is an absolute constant. From (3.3)–(3.5) we get for $\hat{C}C_{\nu}\lambda < 1$

$$\sum_{|T'|=k} |a_{T,T'}| < M\kappa^{m+k} \sum_{p=1}^{\min(k,m)} (m^p/p!) \sum_{\substack{k_1,\ldots,k_p \\ k_1+\ldots+k_p=k}} \sum_{\substack{l_1,\ldots,l_p \\ l_1+\ldots+l_p=m}} (\tilde{C}\lambda)^{m+k-p}$$

$$< M\kappa^{m+k}2^{k+m}(\tilde{C}\lambda)^{\max(k,m)}e^{m} < M\kappa^{m+k}(\bar{D}\lambda)^{\max(m,k)} \tag{3.6}$$

where $\tilde{D}$ is an absolute constant.

Let us consider now the case $m = k$. Following previous arguments, we obtain

$$\sum_{|T'|=k} |\hat{a}_{T,T'}| < \sum_{T:T'\neq T,|T'|=k} \sum_{\pi} \omega_k((t_1, \pi t_1),\dots,(t_k, \pi t_k)) + \sum_{1\leqslant p\leqslant k-1} \sum_{\substack{\tau_1,\dots,\tau_p \\ \cup\tau_i=(T,T')}} \omega_p(\tau_1,\dots,\tau_p)$$

where π in the first term denotes any one-to-one mapping $T \to T'$. Here the first term corresponds to the decompositions of the pair (T, T') onto one-point pairs and the second term corresponds to the remaining decompositions. The second term has a bound similar to the previous case. The first term has the bound

$$M\kappa^{2m}m\left[\sum_{0\neq x\in\mathbb{Z}^{\nu+1}} (\hat{C}\lambda)^{d(x,0)}\right]^{m-1} \sum_{\substack{0\neq x\in\mathbb{Z}^{\nu+1} \\ d(x,0)>1}} (\hat{C}\lambda)^{d(x,0)}$$
$$< M\kappa^{2m}m(D\lambda)^{m-1}\bar{D}\lambda^2 < \tilde{M}\kappa^{2m}(\tilde{D}\lambda)^{m+1}$$

where $\tilde{D}$ is a constant. The lemma is proved.

Proof of Lemma 2.3. By the decomposition

$$A_{11}^k = H_{11}^k + \hat{A}_{11}^k = (H_{11}^k)^{1/2}[E + (H_{11}^k)^{-1/2}\hat{A}_{11}^k(H_{11}^k)^{-1/2}](H_{11}^k)^{1/2} \tag{3.7}$$

we have

$$(A_{11}^k)^{-1} = (H_{11}^k)^{-1/2}(E + V_k)^{-1}(H_{11}^k)^{-1/2}$$

where

$$V_k = (H_{11}^k)^{-1/2}\hat{A}_{11}^k(H_{11}^k)^{-1/2}$$

The operator V_k is clustering and its cluster functions have the property (3.1) for $\lambda = \beta$ and $\kappa = (C\beta)^{-1/2}$, $\hat{M} = M/L$, where M is a constant in (1.6). Moreover, $V_k = \hat{V}_k$.

By (2.3) we get for $k \leqslant N$ and $m, n \leqslant k$

$$\|V_k\|_{m,n} < \hat{M}(C_0\beta)^{-(m+n)/2}\begin{cases}(D\beta)^{\max(m,n)}, & m\neq n\\ (D\beta)^{m+1}, & m=n\end{cases}$$
$$< \hat{M}(C_0D)^{\max(m,n)+1}\begin{cases}\beta^{1/2(m-n)}, & m\neq n\\ \beta, & m=n\end{cases} < K\begin{cases}\beta^{1/2(m-n)}, & m\neq n\\ \beta, & m=n\end{cases}$$

where $K = \hat{M}((C_0^{-1})D)^{N+1}$. Further, for any $p > 1$

$$\|(V_k{}^p)_{m,n}\| \leqslant \sum_{s_1,\dots,p-1} \|(V_k)_{m,s_1}\|\,\|(V_k)_{s_1,s_2}\|\cdots\|(V_k)_{s_{p-1},n}\|$$
$$\leqslant \sum_{s_1,\dots,p-1} a_{m-s_1}a_{s_1-s_2}\cdots a_{s_{p-1}-n} = A_{m-n}^p$$

where

$$a_l = \begin{cases}K\beta^{|l|/2}, & l\neq 0\\ K\beta, & l=0\end{cases}$$

Let us put

$$\varphi(z) = \sum_{k=-\infty}^{\infty} a_k z^k = K\beta + K\beta^{1/2}\left(\frac{z}{1-\beta^{1/2}z} + \frac{1}{z-\beta^{1/2}}\right) \tag{3.8}$$

for $\beta^{1/2} < z < \beta^{-1/2}$.

It is clear that

$$\sum_{k=-\infty}^{\infty} A_k{}^p z^k = (\varphi(z))^p$$

Thus for the series

$$\sum_{p=1}^{\infty} (-1)^p (V_k)^p = G_k \tag{3.9}$$

we have

$$\|(G_k)_{m,n}\| < \sum_{p=1}^{\infty} A_{m-n}^p = B_{m-n}$$

where B_l are the coefficients of the series

$$\varphi(z)/[1-\varphi(z)] = \sum_{l=-\infty}^{\infty} B_l z^l \tag{3.10}$$

if $(h\beta)^{1/2} < z < (h\beta)^{-1/2}$, $h > 1$, is the absolute constant. Thus from (3.10) and (3.8) we get for small β

$$|B_p| < R\beta^{1/2}(\tilde{h}\beta)^{|p|/2}, \qquad p = \pm 1, \pm 2, \ldots \tag{3.11}$$

where R and $\tilde{h}$ are constants. From (3.7) and (3.9) we get

$$(A_{11}^k)^{-1} = (H_{11}^k)^{-1} + (H_{11}^k)^{-1/2} G_k (H_{11}^k)^{-1/2}$$

and by (3.11)

$$\|(A_{11}^k)^{-1}\| < L\beta^{\max(m,n)-m-n}, \qquad F = F(N, M, L)$$

The lemma is proved.

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